On Some Triply Infinite Sums by Means of N-Fractional Calculus(Study on Calculus Operators in Univalent Function Theory)

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数理解析研究所講究録
1538 巻 2007 年 30-43

On Some Triply Infinite Sums by Means of N-Fractional Calculus

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Abstract

In this article some triple infinite sums, some related finite sums and mixed sums, which are derived by means of N- fractional calculus, are reported.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}d\zeta \qquad (\nu \notin \mathbb{Z}^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in \mathbb{Z}^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $\; z \in C$, $\; \nu \in \mathbb{R}$, $\; \Gamma$; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu} = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z), \quad [\textit{Refer to} (1)] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \qquad (m \in Z^{+}), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \qquad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^{\nu}\} = \{N^{\nu} | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f \in F = \{f;\ 0 \neq |f_{\nu}| < \infty, \nu \in R\}$, *where* $f = f(z)$ *and* $z \in C$.
(*vis.* $-\infty < \nu < \infty$).
(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index* ν *" for the set of* F . (*F.O.G.* ; Fractional calculus operator group)

Theorem C. *Let*

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S *is a commutative ring for the function* $f \in F$, *when the identity*

$$N^{\alpha} + N^{\beta} = N^{\gamma} \quad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

(i i) $$(\log(z-c))_{\alpha} = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty),$$

(i i i) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (i i) and (i i i). (Γ; Gamma function),

(i v) $$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. Triply Infinite, Finite and Mixed Sums which are Derived by Means of N- Fractional Calculus

In the following $\alpha, \beta, \gamma, \delta \in R$,

$$\sum_{k,m,n=0}^{s,p,q} \cdots := \sum_{k=0}^{s} \sum_{m=0}^{p} \sum_{n=0}^{q} \cdots , \qquad \sum_{k,m,n=0}^{\infty} \cdots := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cdots ,$$

and

$$\sum_{k,m=0}^{s,p} \cdots := \sum_{k=0}^{s} \sum_{m=0}^{p} \cdots , \qquad \sum_{k,m=0}^{\infty} \cdots := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \cdots ,$$

for our convenience.

We have then Theorem 1. below by the use of N- fractional calculus of products of some power functions.

Theorem 1. *Let*

$$G = G(\alpha, \beta, \gamma ; k, m)$$

$$:= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k! \cdot m! \, \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)} , \qquad (1)$$

$$H = H(\alpha, \gamma, \delta ; k, m, n)$$

$$:= \frac{\Gamma(\delta+1)\Gamma(m+n-\gamma)\Gamma(\gamma+k-\alpha-m+\delta-n)}{n! \, \Gamma(\delta+1-n)\Gamma(m-\gamma)\Gamma(\gamma+k-\alpha-m)} , \qquad (2)$$

and

$$R = R(\alpha, \beta, \gamma, \delta)$$

$$:= -\frac{\sin\pi(\gamma-\alpha-\beta) \cdot \sin\pi(\delta-\alpha)}{\sin\pi(\alpha+\beta) \cdot \sin\pi(\gamma+\delta-\alpha)} , \qquad (3)$$

$$(\, |R| = M < \infty \,) .$$

(i) *When* $\alpha, \beta, \gamma, \delta \notin Z_0^+$, *we have the following triply infinite sums* ;

$$\sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z}\right)^{m+n} \left(\frac{c}{z-c}\right)^{k}$$

$$= R \cdot \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)} \left(\frac{z-c}{z}\right)^{\gamma+\delta-\alpha} , \qquad (4)$$

where

$$|(z-c)/z| \ , \ |c/(z-c)| < 1, \tag{5}$$

$$\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|, \quad \left|\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}\right| < \infty \tag{6}$$

and

$$\left|\frac{\Gamma(\gamma+k-\alpha-m+\delta-n)}{\Gamma(\gamma+k-\alpha-m)}\right|, \quad \left|\frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)}\right| < \infty \ . \tag{7}$$

(ii) *When* $\alpha,\beta,\gamma \notin Z_0^+$ *and* , $\delta = q \in Z^+$ *we have the following mixed sum* ;

$$\sum_{k,m=0}^{\infty} \sum_{n=0}^{q} G\cdot H(\alpha,\gamma,q\,;k,m,n)\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$= R(\alpha,\beta,\gamma,q)\cdot\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(q-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)}\left(\frac{z-c}{z}\right)^{\gamma+q-\alpha}, \tag{8}$$

$$(\,|R(\alpha,\beta,\gamma,q)| = M < \infty\,)$$

having (5), (6) *and*

$$\left|\frac{\Gamma(\gamma+k-\alpha-m+q-n)}{\Gamma(\gamma+k-\alpha-m)}\right| < \infty \ . \tag{9}$$

(iii) *When* $\alpha,\beta \notin Z_0^+$ *and* , $\gamma = p$, $\delta = q$ $(p,q \in Z^+)$ *we have the following mixed sum* ;

$$\sum_{k,m,n=0}^{\infty,\,p,\,q} G(\alpha,\beta,p\,;k,m)\cdot H(\alpha,p,q\,;k,m,n)\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$= \frac{\Gamma(p-\alpha-\beta)\Gamma(q-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)}\left(\frac{z-c}{z}\right)^{p+q-\alpha}, \tag{10}$$

where

$$|(z-c)/z| < \infty, \ |c/(z-c)| < 1. \tag{11}$$

(iv) *When* $\beta \notin Z_0^+$ *and* $\alpha = s$, $\gamma = p$, $\delta = q$ $(s,p,q \in Z^+)$ *we have the following triply finite sum* ;

$$\sum_{k,m,n=0}^{s,p,q} G(s,\beta,p;k,m)\cdot H(s,p,q;k,m,n)\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$=\frac{\Gamma(p-s-\beta)\Gamma(q-s)}{\Gamma(-s-\beta)\Gamma(-s)}\left(\frac{z-c}{z}\right)^{p+q-s}, \qquad (12)$$

where

$$|(z-c)/z|\ ,\ |c/(z-c)|<\infty\ , \qquad (13)$$

and

$$\left|\frac{\Gamma(k-s+p-m)}{\Gamma(k-s)}\right|<\infty\ . \qquad (14)$$

Proof of (i). We have

$$z^{\alpha}=(z-c)^{\alpha}\left(1-\frac{c}{c-z}\right)^{\alpha} \qquad (15)$$

$$=\sum_{k=0}^{\infty}\frac{c^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}(z-c)^{\alpha-k} \qquad (|c/(z-c)|<1) \qquad (16)$$

Make (16) $\times z^{\beta}$, then operate N- fractional calculus operator N^{γ} to its both sides, we obtain

$$(z^{\alpha}\cdot z^{\beta})_{\gamma}=\sum_{k=0}^{\infty}\frac{c^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\left((z-c)^{\alpha-k}\cdot z^{\beta}\right)_{\gamma} \qquad (17)$$

$$=\sum_{k=0}^{\infty}\frac{c^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}\sum_{m=0}^{\infty}\frac{\Gamma(\gamma+1)}{m!\,\Gamma(\gamma+1-m)}\left((z-c)^{\alpha-k}\right)_{\gamma-m}(z^{\beta})_{m}\ , \qquad (18)$$

by Lemma (i v).

Now we have

$$(z^{\alpha}\cdot z^{\beta})_{\gamma}=e^{-i\pi\gamma}P(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}z^{\alpha+\beta-\gamma} \qquad (19)$$

$$\left(\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|<\infty\right),$$

where

$$P(\alpha,\beta,\gamma)=\frac{\sin\pi\alpha\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\alpha)} \tag{20}$$

$$\left(\begin{array}{c} |P(\alpha,\beta,\gamma)|=M<\infty, \\ \mathrm{Re}(\alpha+\beta+1)>0,\ \ (1+\alpha-\gamma)\notin \mathbf{Z}_0^- \end{array}\right)$$

(Refer to J. Frac. Calc. Vol.27, pp.83 - 88) [19].

Next we have

$$\left((z-c)^{\alpha-k}\right)_{\gamma-m}=e^{-i\pi(\gamma-m)}\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}(z-c)^{\alpha-k-\gamma+m}, \tag{21}$$

$$\left(\left|\frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)}\right|<\infty\right)$$

and

$$(z^{\beta})_m=e^{-i\pi m}\frac{\Gamma(m-\beta)}{\Gamma(-\beta)}z^{\beta-m} \tag{22}$$

by Lemma (i), respectively.

We have then

$$\sum_{k,m=0}^{\infty}G(\alpha,\beta,\gamma\ ;k,m)c^k(z-c)^{\alpha-k-\gamma+m}z^{-m}$$

$$=P(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}z^{\alpha-\gamma} \tag{23}$$

$$\left(\left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right|<\infty\right)$$

from (18), (19), (21) and (22).

Make (23) × z^{γ}, then operate N^{δ} to its both sides, we obtain

$$\sum_{k,m=0}^{\infty}G\cdot c^k\left((z-c)^{\alpha-k-\gamma+m}\cdot z^{\gamma-m}\right)_{\delta}=P(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(z^{\alpha-\gamma}\cdot z^{\gamma}\right)_{\delta}, \tag{24}$$

hence

$$\sum_{k,m=0}^{\infty}G\cdot c^k\sum_{n=0}^{\infty}\frac{\Gamma(\delta+1)}{n!\,\Gamma(\delta+1-n)}\left((z-c)^{\alpha-k-\gamma+m}\right)_{\delta-n}(z^{\gamma-m})_n$$

$$=P(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\left(z^{\alpha-\gamma}\cdot z^{\gamma}\right)_{\delta}. \tag{25}$$

Now we have

$$(z^{\alpha-\gamma}\cdot z^{\gamma})_{\delta} = P(\alpha-\gamma,\gamma,\delta)\,(z^{\alpha})_{\delta} \qquad (26)$$

$$= e^{-i\pi\delta}\frac{\sin\pi(\alpha-\gamma)\cdot\sin\pi(\delta-\alpha)}{\sin\pi\alpha\cdot\sin\pi(\delta+\gamma-\alpha)}\cdot\frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)}\,z^{\alpha-\delta} \qquad (27)$$

$$\left(\left|\frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)}\right| < \infty\right)$$

$$\left(\begin{array}{c} |P(\alpha-\gamma,\gamma,\delta)| = M < \infty, \\ \mathrm{Re}(\alpha+1) > 0, \quad (1+\alpha-\gamma-\delta) \notin \mathbf{Z}_0^- \end{array}\right)$$

(Refer to J. Frac. Calc. Vol.27, pp.83 - 88) [19].

Next we have

$$\left((z-c)^{\alpha-k-\gamma+m}\right)_{\delta-n} = e^{-i\pi(\delta-n)}\frac{\Gamma(k+\gamma-\alpha-m+\delta-n)}{\Gamma(k+\gamma-\alpha-m)}(z-c)^{m+\alpha-\gamma-k-\delta+n}, \qquad (28)$$

$$\left(\left|\frac{\Gamma(k+\gamma-\alpha-m+\delta-n)}{\Gamma(k+\gamma-\alpha-m)}\right| < \infty\right)$$

and

$$(z^{\gamma-m})_n = e^{-i\pi n}\frac{\Gamma(m-\gamma+n)}{\Gamma(m-\gamma)}\,z^{\gamma-m-n} \qquad (29)$$

by Lemma (i), respectively.

Therefore, we obtain

$$\sum_{k,m,n=0}^{\infty} G(\alpha,\beta,\gamma\,;k,m)\,H(\alpha,\gamma,\delta\,;k,m,n)\,c^k(z-c)^{\alpha-\gamma-\delta+n+m-k}\,z^{\gamma-m-n}$$

$$= R(\alpha,\beta,\gamma,\delta)\,\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)}\,z^{\alpha-\delta} \qquad (30)$$

from (25) ~ (29), since

$$P(\alpha,\beta,\gamma)\,P(\alpha-\gamma,\gamma,\delta) = R(\alpha,\beta,\gamma,\delta)\ . \qquad (31)$$

We have then (4) from (30), under the conditions, using the notations (1), (2) and (3).

Note 1. When we use

$$(z^{\alpha} z^{\beta})_{\gamma} = (z^{\alpha+\beta})_{\gamma} = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma} \qquad (32)$$

instead of $(z^{\alpha} \cdot z^{\beta})_{\gamma}$ (see Lemma (i v)), we obtain

$$\sum_{k,m=0}^{\infty} G(\alpha,\beta,\gamma\,;k,m) c^{k} (z-c)^{\alpha-k-\gamma+m} z^{-m} = \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha-\gamma}, \qquad (33)$$

instead of (23), from (18).

Therefore, we have the following doubly infinite sum ;

$$\sum_{k,m=0}^{\infty} G \cdot \left(\frac{z-c}{z}\right)^{m} \left(\frac{c}{z-c}\right)^{k} = \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\gamma-\alpha} \qquad (34)$$

from (33).

This result is reported in a previous paper of the author (cf. JFC Vol. 24, (2003), pp.68 - 70.). [11]

When

$$P(\alpha,\beta,\gamma) = 1, \qquad (35)$$

(23) is reduced to (34).

Note 2. When we use

$$(z^{\alpha-\gamma} z^{\gamma})_{\delta} = (z^{\alpha})_{\delta} = e^{-i\pi\delta} \frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)} z^{\alpha-\delta} \qquad \left(\left| \frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)} \right| < \infty \right) \qquad (36)$$

instead of $(z^{\alpha-\gamma} \cdot z^{\gamma})_{\delta}$, we obtain

$$\sum_{k,m,n=0}^{\infty} G \cdot H \cdot c^{k} (z-c)^{m+n-k+\alpha-\gamma-\delta} z^{\gamma-m-n}$$

$$= P(\alpha,\beta,\gamma) \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)} z^{\alpha-\delta} \qquad (37)$$

instead of (30), from (25).

Moreover , for the case of (35) , we have the following triply infinite sum ;

$$\sum_{k,m,n=0}^{\infty} G\cdot H\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)}\left(\frac{z-c}{z}\right)^{\gamma+\delta-\alpha} \tag{38}$$

from (37).

And this result is a special case of (4), in which

$$R(\alpha,\beta,\gamma,\delta)=1. \tag{39}$$

In a previous paper of the author, this result (38) is reported as Theorem 3. in JFC Vol. 24, (2003), p.71 . [11]

Note 3. The identity (4) is same as the one shown in a paper by S. - D. Lin, H. M. Srivastava and S. - T. Tu (cf. JFC Vol. 27, p. 48.) [21].

Proof of (i i). Set $\delta=q\in Z^{+}$ in (4).

Proof of (i i i). Set $\gamma=p,\ \delta=q\ (p,q\in Z^{+})$ in (4).

Proof of (i v). Set $\alpha=s,\ \gamma=p,\ \delta=q\ (s,p,q\in Z^{+})$ in (4).

§ 2. Direct Calculation of Triply Infinite Sum

In the following G, H and R are the ones shown in **§ 1**, respectively.

Now we have

$$\sum_{k,m,n=0}^{\infty} G\cdot H\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$=\sum_{k,m,n=0}^{\infty}\frac{[-\alpha]_k[-\gamma]_m[-\delta]_n[-\beta]_m[m-\gamma]_n}{k!\cdot m!\cdot n!\cdot(-1)^{-k-m-n}}$$

$$\times\frac{\Gamma(k-\alpha+\gamma+\delta-m-n)}{\Gamma(k-\alpha)}\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k} \tag{1}$$

using the relationship

$$\Gamma(\lambda+1-k)=(-1)^{-k}\frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)}\quad, \tag{2}$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \quad [\lambda]_0 = 1. \tag{3}$$

(Notation of Pochhammer).

Next we have

$$\frac{\Gamma(k-\alpha+\gamma+\delta-m-n)}{\Gamma(k-\alpha)} = \frac{[\gamma+\delta-\alpha-m-n]_k}{[-\alpha]_k}\cdot\frac{\Gamma(\gamma+\delta-\alpha-m-n)}{\Gamma(-\alpha)} \tag{4}$$

$$= \frac{[\gamma+\delta-\alpha-m-n]_k}{[-\alpha]_k}\cdot(-1)^{-(m+n)}\frac{\Gamma(\gamma+\delta-\alpha)}{\Gamma(-\alpha)[\alpha-\gamma-\delta+1]_{m+n}} \tag{5}$$

$$= \frac{\Gamma(\gamma+\delta-\alpha)[\gamma+\delta-\alpha-m-n]_k}{\Gamma(-\alpha)[-\alpha]_k}$$

$$\times(-1)^{-(m+n)}\frac{1}{[\alpha-\gamma-\delta+1]_m[m+\alpha-\gamma-\delta+1]_n} \tag{6}$$

since

$$[\alpha-\gamma-\delta+1]_{m+n} = [\alpha-\gamma-\delta+1]_m[m+\alpha-\gamma-\delta+1]_n \ . \tag{7}$$

Therefore, we obtain

$$\sum_{k,m,n=0}^{\infty} G\cdot H\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$= \sum_{k,m,n=0}^{\infty}\frac{\Gamma(\gamma+\delta-\alpha)}{\Gamma(-\alpha)}\cdot\frac{[-\gamma]_m[-\delta]_n[-\beta]_m[m-\gamma]_n}{k!\cdot m!\cdot n!\cdot(-1)^{-k}}$$

$$\times\frac{[\gamma+\delta-\alpha-m-n]_k}{[\alpha-\gamma-\delta+1]_m\,[m+\alpha-\gamma-\delta+1]_n}\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k} \tag{8}$$

from (1) and (6).

Next we have the identity

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k = (1-z)^{-\lambda} \ , \tag{9}$$

hence

$$\sum_{k=0}^{\infty}\frac{[\gamma+\delta-\alpha-m-n]_k}{k!}(-1)^k\left(\frac{c}{z-c}\right)^k=\left(\frac{z}{z-c}\right)^{m+n+\alpha-\gamma-\delta} \quad . \tag{10}$$

Then applying (10) to (8) we obtain

$$\sum_{k,m,n=0}^{\infty} G\cdot H\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^k$$

$$=\frac{\Gamma(\gamma+\delta-\alpha)}{\Gamma(-\alpha)}\left(\frac{z}{z-c}\right)^{\alpha-\gamma-\delta}\sum_{m=0}^{\infty}\frac{[-\gamma]_m[-\beta]_m}{m!\cdot[\alpha-\gamma-\delta+1]_m}$$

$$\times\sum_{n=0}^{\infty}\frac{[-\delta]_n[m-\gamma]_n}{n![m+\alpha-\gamma-\delta+1]_n} \tag{11}$$

$$=\frac{\Gamma(\gamma+\delta-\alpha)\Gamma(\alpha-\gamma-\delta+1)\Gamma(\alpha+\beta+1)}{\Gamma(-\alpha)\Gamma(\alpha-\delta+1)\Gamma(\alpha+\beta-\gamma+1)}\left(\frac{z-c}{z}\right)^{\gamma+\delta-\alpha} , \tag{12}$$

because (see Note 4.)

$$\sum_{n=0}^{\infty}\frac{[-\delta]_n[m-\gamma]_n}{n![m+\alpha-\gamma-\delta+1]_n}={}_2F_1(-\delta, m-\gamma ; m+\alpha-\gamma-\delta+1 ; 1) \tag{13}$$

$$=\frac{\Gamma(m+\alpha-\gamma-\delta+1)\Gamma(\alpha+1)}{\Gamma(m+\alpha-\gamma+1)\Gamma(\alpha+1-\delta)} \qquad \begin{pmatrix}\mathrm{Re}(\alpha+1)>0, \\ (m+\alpha-\gamma-\delta+1)\notin \boldsymbol{Z}_0^-\end{pmatrix} \tag{14}$$

$$=\frac{[\alpha-\gamma-\delta+1]_m}{[\alpha-\gamma+1]_m}\cdot\frac{\Gamma(\alpha-\gamma-\delta+1)\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha+1-\delta)} , \tag{15}$$

and

$$\sum_{m=0}^{\infty}\frac{[-\gamma]_m[-\beta]_m}{m![\alpha-\gamma+1]_m}={}_2F_1(-\gamma, -\beta ; \alpha-\gamma+1; 1) \tag{16}$$

$$=\frac{\Gamma(\alpha-\gamma+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta-\gamma+1)} \qquad \begin{pmatrix}\mathrm{Re}(\alpha+\beta+1)>0, \\ (\alpha-\gamma+1)\notin \boldsymbol{Z}_0^-\end{pmatrix}. \tag{17}$$

Therefore, we obtain

$$\sum_{k,m,n=0}^{\infty} G\cdot H\cdot\left(\frac{z-c}{z}\right)^{m+n}\left(\frac{c}{z-c}\right)^{k}$$

$$= -\frac{\sin\pi(\delta-\alpha)\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma+\delta-\alpha)}\cdot\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)}\left(\frac{z-c}{z}\right)^{\gamma+\delta-\alpha},$$

(§ 1. (4))

from (12), using the relationship

$$\Gamma(\lambda)\Gamma(1-\lambda)=\frac{\pi}{\sin\pi\lambda} \qquad (\lambda\notin \mathbf{Z}). \tag{18}$$

Note 4. We have the following identity ;

$$\sum_{k=0}^{\infty}\frac{[a]_k\,[b]_k}{k!\,[c]_k}= {}_2F_1(a,b;\,c;\,1) \tag{19}$$

$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \begin{pmatrix}\mathrm{Re}(c-a-b)>0,\\ c\notin \mathbf{Z}_0^-\end{pmatrix}. \tag{20}$$

Acknowledgement

The second, third and fourth authors thank to CONDES - University of Zulia for financial support.

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